

Transient Circuit Analysis

RL circuits can be described by first-order differential equations of the form

$$\frac{dx(t)}{dt} + \frac{1}{\tau}x(t) = f(t)$$

where $x(t)$ is some voltage or current in the circuit, and the time constant is $\tau = \frac{L}{R}$.

- If the circuit has no sources, then $f(t) = 0$, and

$$x(t) = Ae^{-t/\tau} \text{ for } t \geq 0$$

where $A = x(0)$.

- If the circuit contains DC sources, then $f(t) = \frac{\lambda}{\tau}$ (a constant), and

$$x(t) = Ae^{-t/\tau} + B \text{ for } t \geq 0$$

where $A = x(0) - \lambda$ and $B = \lambda$. Note also that $\lambda = x(\infty)$.

RC circuits can also be described by first-order differential equations of the form

$$\frac{dx(t)}{dt} + \frac{1}{\tau}x(t) = f(t)$$

where $x(t)$ is a voltage or current in the circuit, but the time constant is now $\tau = RC$.

- If the circuit has no sources, then $f(t) = 0$, and

$$x(t) = Ae^{-t/\tau} \text{ for } t \geq 0$$

where $A = x(0)$.

- If the circuit contains DC sources, then $f(t) = \frac{\lambda}{\tau}$ (a constant), and

$$x(t) = Ae^{-t/\tau} + B \text{ for } t \geq 0$$

where $A = x(0) - \lambda$ and $B = \lambda$. Here again, $\lambda = x(\infty)$.

RLC circuits can be described by second-order differential equations of the form

$$\frac{d^2x(t)}{dt^2} + 2\zeta\omega_n \frac{dx(t)}{dt} + \omega_n^2 x(t) = f(t)$$

and a *characteristic equation* of the form

$$r^2 + 2\zeta\omega_n r + \omega_n^2 = 0.$$

- If the circuit has no sources, then $f(t) = 0$. There are four distinctly different forms possible for the solution, $x(t)$.
 - If $\zeta > 1$, the circuit is said to be **overdamped**, and the characteristic equation has two distinct negative real roots, $r_1 = (-\zeta + \sqrt{\zeta^2 - 1})\omega_n$ and $r_2 = (-\zeta - \sqrt{\zeta^2 - 1})\omega_n$.

The corresponding solution is then

$$x(t) = Ae^{r_1 t} + Be^{r_2 t}$$

where $A = \frac{r_2 x(0) - \dot{x}(0)}{r_2 - r_1}$ and $B = \frac{\dot{x}(0) - r_1 x(0)}{r_2 - r_1}$.

- If $\zeta = 1$, the circuit is said to be **critically damped**, and the characteristic equation has two identical negative roots, $r_1 = r_2 = -\omega_n$. The corresponding solution is then

$$x(t) = (A + Bt)e^{-\omega_n t}$$

where $A = x(0)$ and $B = \dot{x}(0) + \omega_n x(0)$.

- If $1 > \zeta > 0$, the circuit is said to be **underdamped**, and the characteristic equation has two complex conjugate roots, $r_1 = -\zeta\omega_n + j\omega_d$ and $r_2 = -\zeta\omega_n - j\omega_d$, where $\omega_d = \omega_n \sqrt{1 - \zeta^2}$. The corresponding solution is then

$$x(t) = e^{-\zeta\omega_n t} (A \cos \omega_d t + B \sin \omega_d t)$$

where $A = x(0)$ and $B = \frac{\dot{x}(0) + \zeta\omega_n [x(0) - \lambda]}{\omega_d}$.

- If $\zeta = 0$, the circuit is said to be **undamped**, and the characteristic equation has two conjugate imaginary roots, $r_1 = j\omega_n$ and $r_2 = -j\omega_n$. The corresponding solution is then

$$x(t) = A \cos \omega_n t + B \sin \omega_n t$$

where $A = x(0)$ and $B = \frac{\dot{x}(0)}{\omega_n}$.

- If the circuit contains DC sources, then $f(t) = \lambda \omega_n^2$ (a constant). There are again four distinctly different forms possible for the solution, $x(t)$.

- If $\zeta > 1$, the circuit is said to be **overdamped**, and, as before, the characteristic equation has two distinct negative real roots, $r_1 = \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n$ and $r_2 = \left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$. However, the corresponding solution is now

$$x(t) = Ae^{r_1 t} + Be^{r_2 t} + \lambda$$

where $A = \frac{r_2[x(0) - \lambda] - \dot{x}(0)}{r_2 - r_1}$ and $B = \frac{\dot{x}(0) - r_1[x(0) - \lambda]}{r_2 - r_1}$. Note also that

$$\lambda = x(\infty).$$

- If $\zeta = 1$, the circuit is said to be **critically damped**, and, as before, the characteristic equation has two identical negative roots, $r_1 = r_2 = -\omega_n$. The corresponding solution is now

$$x(t) = (A + Bt)e^{-\omega_n t} + \lambda$$

where $A = x(0) - \lambda$ and $B = \dot{x}(0) + \omega_n[x(0) - \lambda]$. Here again, $\lambda = x(\infty)$.

- If $1 > \zeta > 0$, the circuit is said to be **underdamped**, and, as before, the characteristic equation has two complex conjugate roots, $r_1 = -\zeta\omega_n + j\omega_d$ and $r_2 = -\zeta\omega_n - j\omega_d$, where $\omega_d = \omega_n \sqrt{1 - \zeta^2}$. The corresponding solution is now

$$x(t) = e^{-\zeta\omega_n t} (A \cos \omega_d t + B \sin \omega_d t) + \lambda$$

where $A = x(0) - \lambda$ and $B = \frac{\dot{x}(0) + \zeta\omega_n x(0)}{\omega_d}$. Once more, $\lambda = x(\infty)$.

- If $\zeta = 0$, the circuit is said to be *undamped*, and, as before, the characteristic equation has two conjugate imaginary roots, $r_1 = j\omega_n$ and $r_2 = -j\omega_n$. The corresponding solution is now

$$x(t) = A \cos \omega_n t + B \sin \omega_n t + \lambda$$

where $A = x(0) - \lambda$ and $B = \frac{\dot{x}(0)}{\omega_n}$.