Analytical Transient Circuit Analysis Summary

First-Order

1. RL circuits can be described by first-order differential equations of the form

$$\frac{dx(t)}{dt}+\frac{1}{\tau}x(t)=f(t)$$

where $x(t)$ is some voltage or current in the circuit, and the time constant is $\tau=\frac{L}{R}$.

- If the circuit has no sources, then $f(t)=0$, and

$$x(t)=Ae^{-t/\tau} \text{ for } t\geq 0$$

where $A=x(0)$.

- If the circuit contains DC sources, then $f(t)=\frac{\lambda}{\tau}$ (a constant), and

$$x(t)=Ae^{-t/\tau}+B \text{ for } t\geq 0$$

where $A=x(0)-\lambda$ and $B=\lambda$. Note also that $\lambda=x(\infty)$.

2. RC circuits can also be described by first-order differential equations of the form

$$\frac{dx(t)}{dt}+\frac{1}{\tau}x(t)=f(t)$$

where $x(t)$ is a voltage or current in the circuit, but the time constant is now $\tau=RC$.

- If the circuit has no sources, then $f(t)=0$, and

$$x(t)=Ae^{-t/\tau} \quad \text{for } t\geq 0$$

where $A=x(0)$.

- If the circuit contains DC sources, then $f(t)=\frac{\lambda}{\tau}$ (a constant), and

$$x(t) = Ae^{-t/\tau} + B \qquad \text{for } t \geq 0$$

where $A = x(0) - \lambda$ and $B = \lambda$. Here again, $\lambda = x(\infty)$.

Second-Order

RLC circuits can be described by second-order differential equations of the form

$$\frac{d^2x(t)}{dt^2} + 2\zeta\omega_n \frac{dx(t)}{dt} + \omega_n^2 x(t) = f(t)$$

and a *characteristic equation* of the form

$$r^2 + 2\zeta\omega_n r + \omega_n^2 = 0.$$

- If the circuit has no sources, then $f(t) = 0$. There are four distinctly different forms possible for the solution, $x(t)$:
  - If $\zeta > 1$, the circuit is said to be ***overdamped***, and the characteristic equation has two distinct negative real roots, $r_1 = \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n$ and $r_2 = \left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$. The corresponding solution is then

    $$x(t) = Ae^{r_1 t} + Be^{r_2 t}$$

    where $A = \dfrac{r_2 x(0) - \dot{x}(0)}{r_2 - r_1}$ and $B = \dfrac{\dot{x}(0) - r_1 x(0)}{r_2 - r_1}$.

  - If $\zeta = 1$, the circuit is said to be ***critically damped***, and the characteristic equation has two identical negative real roots, $r_1 = r_2 = -\omega_n$. The corresponding solution is then

    $$x(t) = (A + Bt)e^{-\omega_n t}$$

    where $A = x(0)$ and $B = \dot{x}(0) + \omega_n x(0)$.

  - If $1 > \zeta > 0$, the circuit is said to be ***underdamped***, and the characteristic equation has two complex conjugate roots, $r_1 = -\zeta\omega_n + j\omega_d$ and $r_2 = -\zeta\omega_n - j\omega_d$, where $\omega_d = \omega_n\sqrt{1 - \zeta^2}$. The corresponding solution is then

    $$x(t) = e^{-\zeta\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right)$$

where $A = x(0)$ and $B = \dfrac{\dot{x}(0) + \zeta\omega_n\left[x(0) - \lambda\right]}{\omega_d}$.

- If $\zeta = 0$, the circuit is said to be ***undamped***, and the characteristic equation has two conjugate imaginary roots, $r_1 = j\omega_n$ and $r_2 = -j\omega_n$. The corresponding solution is then

$$x(t) = A\cos\omega_n t + B\sin\omega_n t$$

  where $A = x(0)$ and $B = \dfrac{\dot{x}(0)}{\omega_n}$.

- If the circuit contains DC sources, then $f(t) = \lambda\omega_n^2$ (a constant). There are again four distinctly different forms possible for the solution, $x(t)$.

  - If $\zeta > 1$, the circuit is said to be ***overdamped***, and, as before, the characteristic equation has two distinct negative real roots, $r_1 = \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n$ and $r_2 = \left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$. However, the corresponding solution is now

$$x(t) = Ae^{r_1 t} + Be^{r_2 t} + \lambda$$

    where $A = \dfrac{r_2\left[x(0) - \lambda\right] - \dot{x}(0)}{r_2 - r_1}$ and $B = \dfrac{\dot{x}(0) - r_1\left[x(0) - \lambda\right]}{r_2 - r_1}$. Note also that $\lambda = x(\infty)$.

  - If $\zeta = 1$, the circuit is said to be ***critically damped***, and, as before, the characteristic equation has two identical negative real roots, $r_1 = r_2 = -\omega_n$. The corresponding solution is now

$$x(t) = (A + Bt)e^{-\omega_n t} + \lambda$$

    where $A = x(0) - \lambda$ and $B = \dot{x}(0) + \omega_n\left[x(0) - \lambda\right]$. Here again, $\lambda = x(\infty)$.

  - If $1 > \zeta > 0$, the circuit is said to be ***underdamped***, and, as before, the characteristic equation has two complex conjugate roots, $r_1 = -\zeta\omega_n + j\omega_d$ and $r_2 = -\zeta\omega_n - j\omega_d$, where $\omega_d = \omega_n\sqrt{1 - \zeta^2}$. The corresponding solution is now

$$x(t) = e^{-\zeta\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right) + \lambda$$

where $A = x(0) - \lambda$ and $B = \dfrac{\dot{x}(0) + \zeta\omega_n x(0)}{\omega_d}$. Once more, $\lambda = x(\infty)$.

- If $\zeta = 0$, the circuit is said to be ***undamped***, and, as before, the characteristic equation has two conjugate imaginary roots, $r_1 = j\omega_n$ and $r_2 = -j\omega_n$. The corresponding solution is now

$$x(t) = A\cos\omega_n t + B\sin\omega_n t + \lambda$$

where $A = x(0) - \lambda$ and $B = \dfrac{\dot{x}(0)}{\omega_n}$.