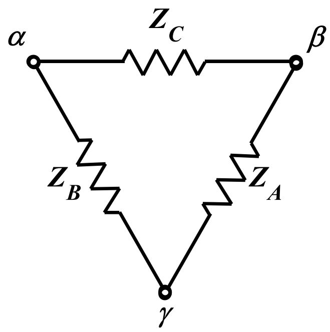
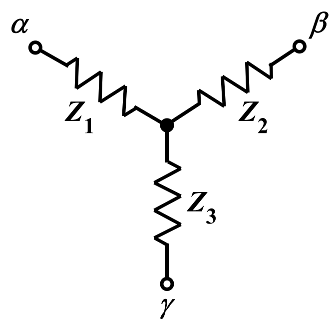
Delta-to-Wye and Wye-to-Delta Transformations

In the process of circuit analysis we often encounter three-terminal subcircuits such as those shown below. Many times it would simplify the analysis procedure if we could convert the “delta” (also called “pi”) connected circuit to an equivalent “wye” (also called “tee”) configuration, or conversely, convert the wye to an equivalent delta.



Delta-Connected Impedances



Wye-Connected Impedances

The transformation equations are shown below. The objective of this note is to derive those equations and therefore show that the transformations do indeed result in equivalent subcircuits – that is, we can replace one subcircuit with the other, and it will be impossible to determine any difference in behavior with respect to the terminal connections; they behave identically.

$\Delta \rightarrow Y$ Transformation

Converting from a delta configuration to a wye configuration is accomplished as follows:

$$Z_1 = \frac{Z_B Z_C}{Z_A + Z_B + Z_C}$$

$$Z_2 = \frac{Z_A Z_C}{Z_A + Z_B + Z_C}$$

$$Z_3 = \frac{Z_A Z_B}{Z_A + Z_B + Z_C}$$

$\mathbf{Y \rightarrow \Delta}$ Transformation

Similarly, a wye-connected subcircuit can be easily converted to a delta-connected configuration as follows:

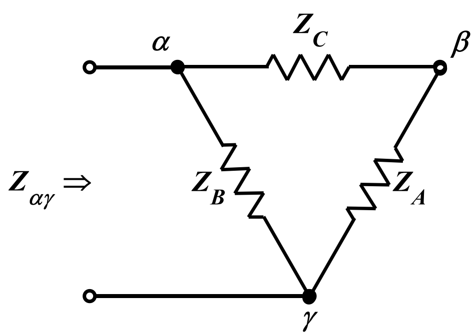
$$Z_A = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}{Z_1}$$

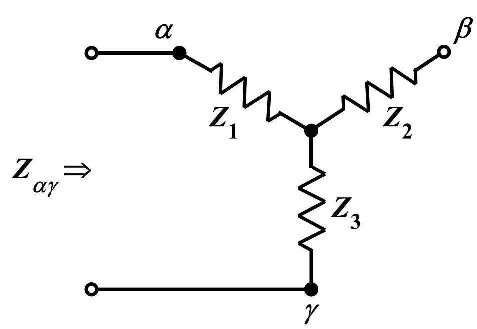
$$Z_B = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}{Z_2}$$

$$Z_C = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3}{Z_3}$$

How Can We Derive the Transformation Equations?

Consider the two diagrams shown below. The equivalent impedance with respect to the $\alpha - \gamma$ terminal pair must be the same for both circuits if they are indeed equivalent.



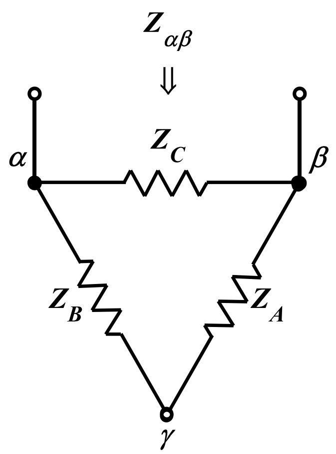


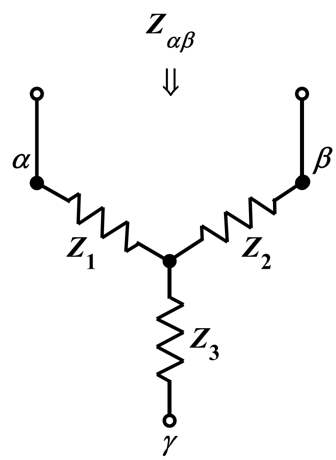
$$Z_{\alpha\gamma} = Z_B \| (Z_A + Z_C) = \frac{Z_A Z_B + Z_B Z_C}{Z_A + Z_B + Z_C} \quad \text{and} \quad Z_{\alpha\gamma} = Z_1 + Z_3$$

So,

$$Z_1 + Z_3 = \frac{Z_A Z_B + Z_B Z_C}{Z_A + Z_B + Z_C} \tag{1}$$

Similarly,



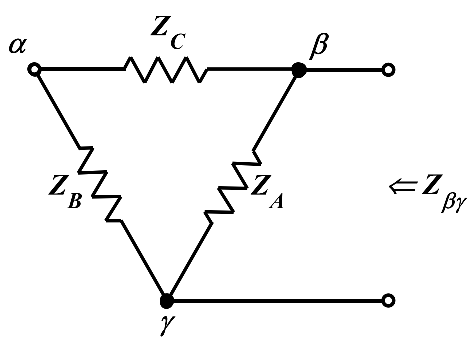


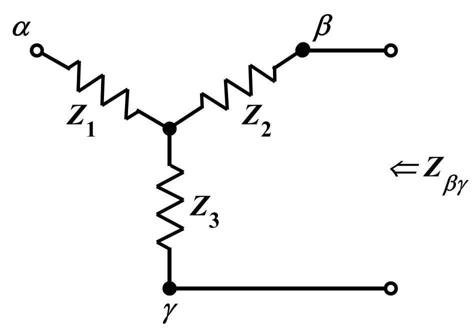
$$Z_{\alpha\beta} = Z_C \| (Z_A + Z_B) = \frac{Z_A Z_C + Z_B Z_C}{Z_A + Z_B + Z_C} \qquad \text{and} \qquad Z_{\alpha\beta} = Z_1 + Z_2$$

So,

$$Z_1 + Z_2 = \frac{Z_A Z_C + Z_B Z_C}{Z_A + Z_B + Z_C} \tag{2}$$

And, for the two diagrams shown below:





$$Z_{\beta\gamma} = Z_A \| (Z_B + Z_C) = \frac{Z_A Z_B + Z_A Z_C}{Z_A + Z_B + Z_C} \qquad \text{and} \qquad Z_{\beta\gamma} = Z_2 + Z_3$$

So,

$$Z_2 + Z_3 = \frac{Z_A Z_B + Z_A Z_C}{Z_A + Z_B + Z_C} \tag{3}$$

Then, $(1)+(2)-(3)$ gives

$$(Z_1 + Z_3) + (Z_1 + Z_2) - (Z_2 + Z_3) = \frac{(Z_A Z_B + Z_B Z_C) - (Z_A Z_C + Z_B Z_C) + (Z_A Z_B + Z_A Z_C)}{Z_A + Z_B + Z_C}$$

or $$2Z_1 = \frac{2Z_B Z_C}{Z_A + Z_B + Z_C} \quad \Rightarrow \quad Z_1 = \frac{Z_B Z_C}{Z_A + Z_B + Z_C}$$

Similarly, $-(1)+(2)+(3)$ gives

$$-(Z_1 + Z_3) + (Z_1 + Z_2) + (Z_2 + Z_3) = \frac{-(Z_A Z_B + Z_B Z_C) + (Z_A Z_C + Z_B Z_C) + (Z_A Z_B + Z_A Z_C)}{Z_A + Z_B + Z_C}$$

or $$2Z_2 = \frac{2Z_A Z_C}{Z_A + Z_B + Z_C} \quad \Rightarrow \quad Z_2 = \frac{Z_A Z_C}{Z_A + Z_B + Z_C}$$

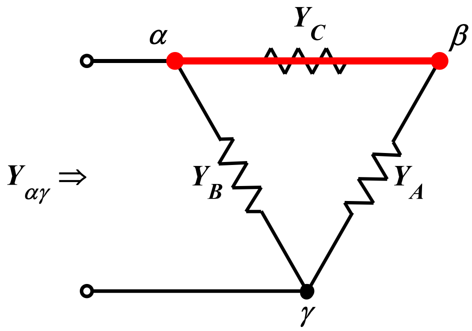
and $(1)-(2)+(3)$ yields

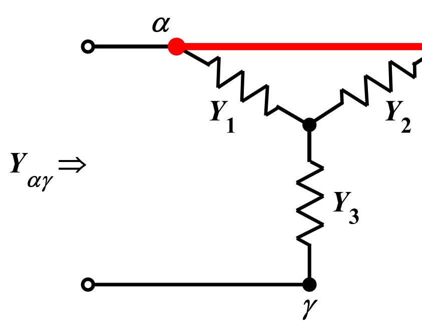
$$(Z_1 + Z_3) - (Z_1 + Z_2) + (Z_2 + Z_3) = \frac{(Z_A Z_B + Z_B Z_C) - (Z_A Z_C + Z_B Z_C) + (Z_A Z_B + Z_A Z_C)}{Z_A + Z_B + Z_C}$$

or $$2Z_3 = \frac{2Z_A Z_B}{Z_A + Z_B + Z_C} \quad \Rightarrow \quad Z_3 = \frac{Z_A Z_B}{Z_A + Z_B + Z_C}$$

This verifies the first set of transformation equations.

Now, consider the following: (The red lines designate a short circuit between the two terminals.)



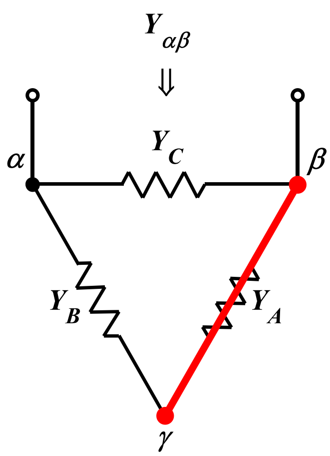


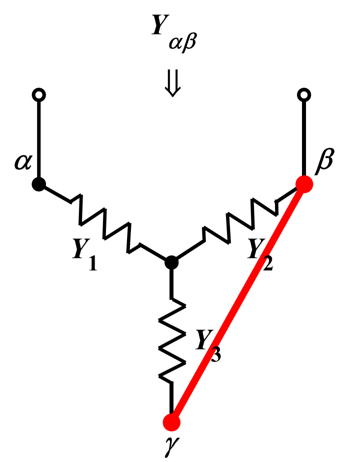
$$Y_{\alpha\gamma} = Y_A + Y_B \qquad \text{and} \qquad Y_{\alpha\gamma} = \frac{(Y_1 + Y_2)Y_3}{Y_1 + Y_2 + Y_3}$$

So,

$$Y_A + Y_B = \frac{Y_1 Y_3 + Y_2 Y_3}{Y_1 + Y_2 + Y_3} \tag{4}$$

Similarly



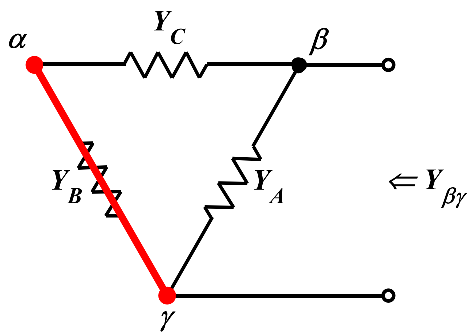


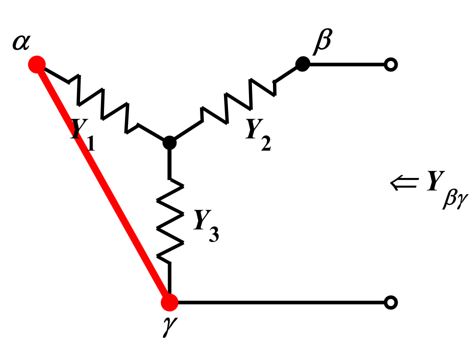
$$Y_{\alpha\beta} = Y_B + Y_C \qquad \text{and} \qquad Y_{\alpha\beta} = \frac{(Y_2 + Y_3)Y_1}{Y_1 + Y_2 + Y_3}$$

So,

$$Y_B + Y_C = \frac{Y_1 Y_2 + Y_1 Y_3}{Y_1 + Y_2 + Y_3} \tag{5}$$

and





$Y_{\beta\gamma} = Y_A + Y_C$ and $Y_{\beta\gamma} = \dfrac{(Y_1 + Y_3)Y_2}{Y_1 + Y_2 + Y_3}$

So, $$Y_A + Y_C = \frac{Y_1Y_2 + Y_2Y_3}{Y_1 + Y_2 + Y_3} \qquad (6)$$

Now $(4)-(5)+(6)$ gives

$$(Y_A + Y_B) - (Y_B + Y_C) + (Y_A + Y_C) = \frac{(Y_1Y_3 + Y_2Y_3) - (Y_1Y_2 + Y_1Y_3) + (Y_1Y_2 + Y_2Y_3)}{Y_1 + Y_2 + Y_3}$$

or $$2Y_A = \frac{2Y_2Y_3}{Y_1 + Y_2 + Y_3} \quad \Rightarrow \quad Y_A = \frac{Y_2Y_3}{Y_1 + Y_2 + Y_3}$$

Therefore, $$Z_A = \frac{Y_1 + Y_2 + Y_3}{Y_2Y_3} = \frac{Z_2Z_3 + Z_1Z_3 + Z_1Z_2}{Z_1} \text{ or } Z_A = Z_2 + Z_3 + \frac{Z_2Z_3}{Z_1}$$

$(4)+(5)-(6)$ gives

$$(Y_A + Y_B) + (Y_B + Y_C) - (Y_A + Y_C) = \frac{(Y_1Y_3 + Y_2Y_3) + (Y_1Y_2 + Y_1Y_3) - (Y_1Y_2 + Y_2Y_3)}{Y_1 + Y_2 + Y_3}$$

or $$2Y_B = \frac{2Y_1Y_3}{Y_1 + Y_2 + Y_3} \quad \Rightarrow \quad Y_B = \frac{Y_1Y_3}{Y_1 + Y_2 + Y_3}$$

Therefore, $$Z_B = \frac{Y_1 + Y_2 + Y_3}{Y_1Y_3} = \frac{Z_2Z_3 + Z_1Z_3 + Z_1Z_2}{Z_2} \text{ or } Z_B = Z_1 + Z_3 + \frac{Z_1Z_3}{Z_2}$$

and $-(4)+(5)+(6)$ gives

$$-\left(Y_A+Y_B\right)+\left(Y_B+Y_C\right)+\left(Y_A+Y_C\right)=\frac{-\left(Y_1Y_3+Y_2Y_3\right)+\left(Y_1Y_2+Y_1Y_3\right)+\left(Y_1Y_2+Y_2Y_3\right)}{Y_1+Y_2+Y_3}$$

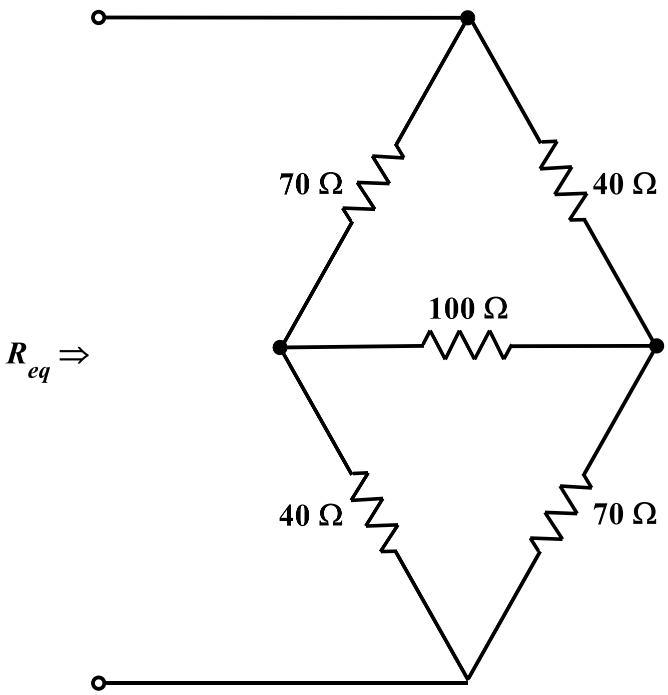
or $$2Y_C=\frac{2Y_1Y_2}{Y_1+Y_2+Y_3} \qquad \Rightarrow \qquad Y_C=\frac{Y_1Y_2}{Y_1+Y_2+Y_3}$$

Therefore, $$Z_C=\frac{Y_1+Y_2+Y_3}{Y_1Y_2}=\frac{Z_2Z_3+Z_1Z_3+Z_1Z_2}{Z_3} \text{ or } Z_C=Z_1+Z_2+\frac{Z_1Z_2}{Z_3}$$

These results verify the second set of transformation equations.

Example 1:

Determine the equivalent resistance of the circuit shown.



As is, determining the equivalent resistance is very difficult. However, by applying the $\mathbf{\Delta \rightarrow Y}$ transformation equations discussed earlier, the problem can be greatly simplified. Looking at the bottom half of the circuit, and comparing with the notes above:

$Z_A = 600\ \Omega$, $Z_B = 300\ \Omega$, and $Z_C = 100\ \Omega$.

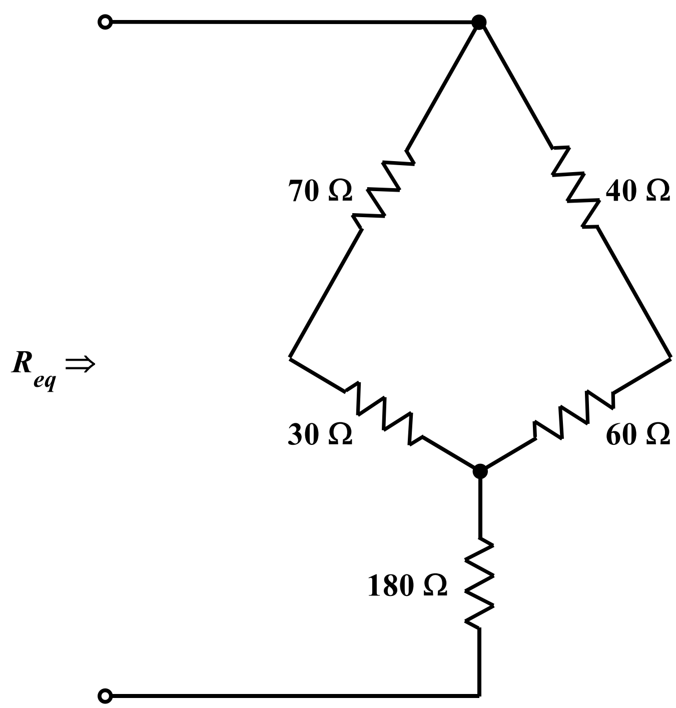
Applying the $\mathbf{\Delta \rightarrow Y}$ transformation equations:

$$Z_1 = \frac{Z_B Z_C}{Z_A + Z_B + Z_C} = \frac{(300\ \Omega)(100\ \Omega)}{(600\ \Omega)+(300\ \Omega)+(100\ \Omega)} = \frac{30,000}{1000}\ \Omega = 30\ \Omega$$

$$Z_2 = \frac{Z_A Z_C}{Z_A + Z_B + Z_C} = \frac{(600\ \Omega)(100\ \Omega)}{(600\ \Omega)+(300\ \Omega)+(100\ \Omega)} = \frac{60,000}{1000}\ \Omega = 60\ \Omega$$

$$Z_3 = \frac{Z_A Z_B}{Z_A + Z_B + Z_C} = \frac{(600\ \Omega)(300\ \Omega)}{(600\ \Omega)+(300\ \Omega)+(100\ \Omega)} = \frac{180,000}{1000}\ \Omega = 180\ \Omega$$

Now, we have the following equivalent circuit:

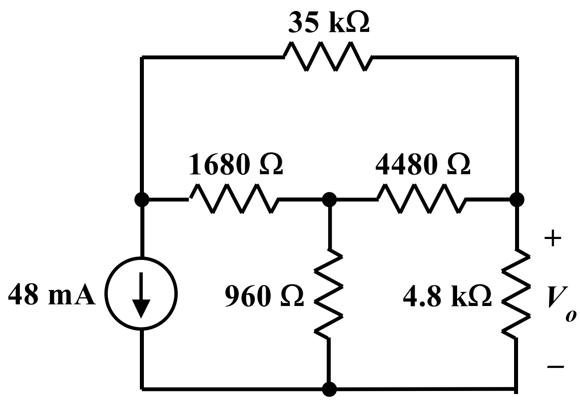


Here, determining the equivalent resistance is straightforward:

$$\begin{aligned} R_{eq} &= \{[(70\ \Omega)+(30\ \Omega)] \| [(40\ \Omega)+(60\ \Omega)]\} + (180\ \Omega) \\ &= [(100\ \Omega) \| (100\ \Omega)] + (180\ \Omega) \\ &= (50\ \Omega) + (180\ \Omega) \\ &= 230\ \Omega \end{aligned}$$

Example 2:

Determine the value of V_o in the circuit shown.



The three resistors in the middle of the circuit comprise a "wye" subcircuit, with $Z_1 = 1680\ \Omega$, $Z_2 = 4480\ \Omega$, and $Z_3 = 960\ \Omega$.

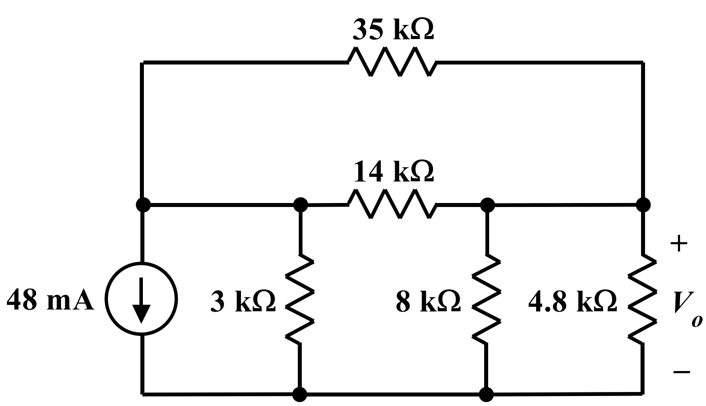
Applying the $\mathbf{Y} \to \mathbf{\Delta}$ transformation equations:

$$\begin{aligned} Z_A &= \frac{Z_1Z_2 + Z_2Z_3 + Z_1Z_3}{Z_1} \\ &= \frac{(1680\ \Omega)(4480\ \Omega) + (4480\ \Omega)(960\ \Omega) + (1680\ \Omega)(960\ \Omega)}{1680\ \Omega} \\ &= \frac{13,440,000}{1680}\ \Omega = 8\ \text{k}\Omega \end{aligned}$$

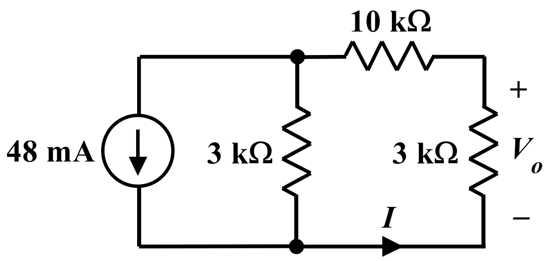
$$Z_B = \frac{Z_1Z_2 + Z_2Z_3 + Z_1Z_3}{Z_2} = \frac{13,440,000}{4480}\ \Omega = 3\ \text{k}\Omega$$

$$Z_C = \frac{Z_1Z_2 + Z_2Z_3 + Z_1Z_3}{Z_3} = \frac{13,440,000}{960}\ \Omega = 14\ \text{k}\Omega$$

Now we have the following equivalent circuit:



Now, note that the $35\ \text{k}\Omega$ and $14\ \text{k}\Omega$ resistors are in parallel and equivalent to $\frac{(35\ \text{k}\Omega)(14\ \text{k}\Omega)}{(35\ \text{k}\Omega)+(14\ \text{k}\Omega)} = 10\ \text{k}\Omega$. Also, the $8\ \text{k}\Omega$ and $4.8\ \text{k}\Omega$ resistors are in parallel and equivalent to $\frac{(8\ \text{k}\Omega)(4.8\ \text{k}\Omega)}{(8\ \text{k}\Omega)+(4.8\ \text{k}\Omega)} = 3\ \text{k}\Omega$. Thus, the circuit can be redrawn as follows;



Hence, the circuit can be treated as a current divider with

$$I = \frac{3\ \text{k}\Omega}{(3\ \text{k}\Omega)+\left[(3\ \text{k}\Omega)+(10\ \text{k}\Omega)\right]}\cdot(48\ \text{mA}) = \frac{3}{16}\cdot(48\ \text{mA}) = 9\ \text{mA}$$

and then

$$V_o = -(3\ \text{k}\Omega)\cdot I = -27\ \text{V}$$